

CONVERGENT NORMAL FORM FOR REAL HYPERSURFACES AT GENERIC LEVI DEGENERACY

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ABSTRACT. We construct a complete convergent normal form for a real hypersurface in \mathbb{C}^N , $N \geq 2$ at generic Levi degeneracy. This seems to be the first convergent normal form for a Levi-degenerate hypersurface. In particular, we obtain, in the spirit of the work of Chern and Moser [5], distinguished curves in the Levi degeneracy set, that we call *degenerate chains*.

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1. INTRODUCTION

In the study of any geometric structure on a manifold M , a *normal form*, corresponding to special choices of coordinates adapted to the structure, is of fundamental importance. In case of Riemannian metric, such special coordinates are the normal coordinates given by the exponential map. For a vector field X that does not vanish at a point $p \in M$, special local coordinates can be chosen, where this vector field is constant. If, however, X does vanish at p , its normal form, known as *Poincare-Dulac normal form* [10], exists in general only in formal sense, whereas its convergence is a delicate issue depending on presence of so-called small divisors. Because of such clear difference in behavior, a point where X vanishes, is to be treated as *singularity of X* .

The study of real submanifolds M in complex spaces \mathbb{C}^n is remarkable in which it exhibits both regular and singular phenomena. A simple example of a regular point is a *CR point* $p_0 \in M$, for which the *complex tangent space*

$$T_p^{\mathbb{C}} := T_p M \cap J T_p M$$

is of constant dimension for $p \in M$ near p_0 , whereas a point p_0 for which this dimension is not constant in any neighborhood, is called a *CR singularity*. (Here J is the standard complex

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structure on \mathbb{C}^n .) In relation with normal form (and many other questions), a regular behavior occurs for *Levi-nondegenerate hypersurfaces*, where a normal form was constructed by Chern and Moser [5]. On the other hand, at “singular” points, where the Levi form is degenerate, no such normal form is known in general.

A particular property of the normal form constructed by Chern and Moser is its *convergence* for M real-analytic. As a consequence, geometry of the CR structure of M can be studied using its normal form. Another remarkable consequence is the presense of so-called *chains*, i.e. certain distinguished real curves that can be locally constructed as certain lines in coordinates corresponding to the normal form.

Since the work of Chern and Moser, normal forms have been constructed *in the formal sense* for certain classes of Levi-degenerate hypersurfaces by Ebenfelt [7, 6], Wong [16], Kolar [11], Kolar and Lamel [13]. We also refer to Moser-Webster [14], Huang and Yin [9], and Burcea [4] for normal forms at CR-singular points. These forms were either known to be divergent in general or the question about their convergence was left open. See e.g. [14] where convergence only holds in so-called *elliptic case* or [12] for examples of divergent normal forms. For normal forms for Levi-nondegenerate CR-manifolds in higher codimension see Ezhov and Schmalz [8] and Beloshapka [3].

The goal of this paper is to establish what seems to be the first convergent normal form for Levi-degenerate real-analytic hypersurfaces. Our normal form is constructed for the well-known class of *generic Levi-degeneracy* points introduced by Webster [15], i.e. points where the determinant of the Levi form vanishes but its differential restricted to the complex tangent space doesn't. Note that a point of generic Levi degeneracy is “stable” in the sense that it cannot be removed by a small perturbation. A different *formal normal form* for generic Levi degeneracy points, whose convergence remains unknown, was constructed by Ebenfelt [6]. More recently, Kolar [11] constructed a formal normal form for all hypersurfaces of finite type in \mathbb{C}^2 , which, however, is divergent in general (see [12]).

The convergence proof for the normal form by Chern and Moser is heavily based on the property that Levi forms at different points are isomorphic. In fact, the geometry and normal forms look similar at all points. Consequently, normalization conditions for the normal form at a point $p \in M$ depend analytically on p , and hence, can be translated into systems of certain analytic ordinary differential equations whose solutions are again real-analytic. This is *not the case any more* for points of generic Levi degeneracy, in whose neighborhoods the Levi form does not have constant rank. Thus geometry at those degenerate points is fundamentally different from that at Levi-nondegenerate points. To overcome this new difficulty, we first restrict to the subset of all Levi-degenerate points of M which, at a point of generic Levi degeneracy, is always a (real-analytic) hypersurface Σ in M , transverse to the Levi kernel at each point of it. Then we set up a system of real-analytic ordinary differential equations *along* Σ defining analogues of Chern-Moser chains, that we call *degenerate chains*, and further differential equations to restrict (normalize) parametrizations of degenerate chains. Through every point of Σ we obtain a unique (in the sense of germs) degenerate chain, and its normalized parametrization is determined up to linear scaling. Also note that our degenerate chains can be still defined for merely smooth real hypersurfaces by the same equations.

We now give complete statements of our normal form, which are different in \mathbb{C}^2 and higher dimension. This is due to the presense of Levi-nondegenerate directions in higher dimension. We

write

$$(z, w) = (z, u + iv) \in \mathbb{C} \times \mathbb{C}, \quad (\check{z}, z_n, w) = (z_1, \dots, z_{n-1}, z_n, u + iv) \in \mathbb{C}^{n-1} \times \mathbb{C} \times \mathbb{C},$$

for the coordinates in \mathbb{C}^2 and in \mathbb{C}^{n+1} , $n \geq 2$, respectively. Then in \mathbb{C}^2 we obtain:

Theorem 1. *Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface with generic Levi degeneracy at a point $p \in M$. Then there exists a local biholomorphic transformation F of \mathbb{C}^2 sending p to 0 and M into a normal form*

$$v = 2\operatorname{Re}(z^2 \bar{z}) + \sum_{k,l \geq 2} \Phi_{kl}(u) z^k \bar{z}^l, \quad \operatorname{Re} \Phi_{32}(u) = \operatorname{Im} \Phi_{42}(u) = 0. \quad (1.1)$$

The germ of F at p is uniquely determined by the restriction of its differential dF to the complex tangent space $T_p^{\mathbb{C}} M$.

Furthermore, the Levi degeneracy set of M is canonically foliated by degenerate chains, where the chain through p is locally given by $z = 0$ in any normal form at p .

The normal form here is similar but different from the formal normal form of Kolar [11].

In the case of higher dimension, we write M as

$$v = \sum_{k', k, l', l \in \mathbb{Z}_{\geq 0}} \Phi_{k'kl'l}(\check{z}, \bar{\check{z}}, u) z_n^k \bar{z}_n^l,$$

where each $\Phi_{k'kl'l}(\check{z}, \bar{\check{z}}, u) \in \mathbb{C}\{u\}[\check{z}, \bar{\check{z}}]$ is a bi-homogeneous polynomial of bi-degree (k', l') in $(\check{z}, \bar{\check{z}})$ with real-analytic coefficients in u . Denote by $r \geq \frac{n-1}{2}$ the number of positive eigenvalues of the Levi form of M at a reference point $p \in M$, and write

$$\langle \check{z}, \bar{\check{z}} \rangle := \sum_{j=1}^{n-1} \varepsilon^j |z_j|^2, \quad \varepsilon^j = 1 \text{ for } j \leq r, \quad \varepsilon^j = -1 \text{ for } j > r, \quad (1.2)$$

and

$$\mathbf{tr} := \sum_{j=1}^{n-1} \varepsilon^j \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$$

for the *trace operator* (that is also used by Chern and Moser in their normal form). Now our main result in higher dimension is the following:

Theorem 2. *Let $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, be a real-analytic hypersurface with generic Levi degeneracy at a point $p \in M$. Then there exists a local biholomorphic transformation F of \mathbb{C}^{n+1} sending p into 0 and M into a normal form*

$$v = \langle \check{z}, \bar{\check{z}} \rangle + 2\operatorname{Re}(z_n^2 \bar{z}_n) + \sum_{k' \geq 2} (\Phi_{k'001}(\check{z}, \bar{\check{z}}, u) \bar{z}_n + \Phi_{01k'0}(\check{z}, \bar{\check{z}}, u) z_n) + \sum_{k'+k, l'+l \geq 2} \Phi_{k'kl'l}(\check{z}, \bar{\check{z}}, u) z_n^k \bar{z}_n^l,$$

where terms in the last sum satisfy the normalization conditions

$$\begin{aligned} \Phi_{1102} &= 0, \quad \mathbf{tr} \Phi_{1111} = 0, \\ \operatorname{Re} \Phi_{1211} &= 0, \quad \operatorname{Im} \mathbf{tr} \Phi_{1211} = 0. \end{aligned} \quad (1.3)$$

The germ of F at p is uniquely determined by the restriction of its differential dF to the complex tangent space $T_p^{\mathbb{C}} M$, which in turn, is uniquely determined by a pair (λ, C) , where λ is a real scalar and C is a complex-linear automorphism of \mathbb{C}^{n-1} such that

$$\langle C\check{z}, \overline{C\check{z}} \rangle = \lambda \langle \check{z}, \bar{\check{z}} \rangle.$$

Furthermore, the Levi degeneracy set of M is canonically foliated by degenerate chains, where the chain through p is locally given by $z = 0$ in any normal form at p .

Note that our normal form here is very different from [6], and also depends on fewer parameters. In fact, the normal form is parametrized precisely by the group of all automorphisms (fixing the origin) of the model hypersurface

$$v = \langle \bar{z}, \bar{z} \rangle + 2\operatorname{Re}(z_n^2 \bar{z}_n), \quad (1.4)$$

which, as a consequence of Theorem 2, consists of all linear automorphisms of \mathbb{C}^{n+1} , preserving (1.4).

2. NOTATIONS AND PRELIMINARIES

Recall that the *Levi form* of a real hypersurface $M \subset \mathbb{C}^{n+1}$, $n \geq 1$ is defined for CR points p by

$$\mathcal{L}_p: T_p^{\mathbb{C}}M \times T_p^{\mathbb{C}}M \rightarrow \mathbb{C} \otimes (T_p M / T_p^{\mathbb{C}}M), \quad \mathcal{L}_p(X(p), Y(p)) = [X^{10}, Y^{01}](p) \mod \mathbb{C} \otimes T_p M,$$

where X and Y are vector fields in $T^{\mathbb{C}}M$ and

$$X^{10} := X - iJX, \quad X^{01} := X + iJX,$$

are the corresponding $(1, 0)$ and $(0, 1)$ vector fields.

For a real-analytic hypersurface $M \subset \mathbb{C}^2$ with generic Levi degeneracy at a point $p \in M$, we consider its Levi degeneracy set Σ , which for $M \subset \mathbb{C}^2$ is simply the set of points where the whole Levi form is zero. In our case of generic Levi degeneracy Σ is a totally real submanifold. Then for $p \in \Sigma$ we can naturally define, using third order Lie brackets, the canonical cubic form

$$c: (T_p^{\mathbb{C}}M)^3 \rightarrow \mathbb{C} \otimes \frac{T_p M}{T_p^{\mathbb{C}}M}, \quad c(X(p), Y(p), Z(p)) = [X^{10}, [Y^{10}, Z^{01}]](p) \mod \mathbb{C} \otimes T_p^{\mathbb{C}}M. \quad (2.1)$$

We can always choose local coordinates in \mathbb{C}^2 in such a way that

$$p = 0, \quad T_0 M = \{v = 0\}, \quad T_0 \Sigma = \{v = 0, \operatorname{Re} z = 0\},$$

and the canonical cubic form at 0 is given by $z^2 \bar{z}$. In what follows we always assume that coordinates for M are chosen in this manner. Considering a hypersurface, given near the origin by the defining equation $v = \Phi(z, \bar{z}, u)$, and using expansions of the form

$$\Phi(z, \bar{z}, u) = \sum_{k, l \geq 0} \Phi_{kl}(u) z^k \bar{z}^l,$$

where $\Phi_{kl}(u)$ are real-analytic near the origin, we can read the above normalization of the canonical cubic form as $\Phi_{21}(0) = 1$.

Let then $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, be a real-analytic hypersurface with generic Levi degeneracy at a point $p \in M$. It is shown in [6] that one can define for M a third-order invariant cubic form

$$c: T_p^{10}M \times T_p^{10}M \times K_p^{01} \rightarrow \mathbb{C} \otimes \frac{T_p M}{T_p^{\mathbb{C}}M}, \quad (2.2)$$

which we call *Ebenfelt's tensor*, such that its restriction onto $(K_p^{10})^2 \times K_p^{01}$ is non-vanishing (here $K_p \subset T_p^{\mathbb{C}}M$ is the Levi kernel of M at p , $\dim_{\mathbb{C}} K_p = 1$). The canonical cubic form (2.2)

replaces the tensor (2.1) in the higher dimensional case and enables us to fix canonically the *Levi-nondegenerate direction at p* as the complex hyperplane

$$K_p^T = \left\{ X \in T_p^{\mathbb{C}} M : c(X^{10}, Y^{10}, Z^{01}) = 0 \quad \forall Y, Z \in K_p \right\} \subset T_p^{\mathbb{C}} M.$$

The subspace K_p^T is transverse to K_p . We assume in what follows that local coordinates for M near p are chosen in such a way that p is the origin, the tangent space at 0 is $\{v = 0\}$, the Levi kernel at 0 is given by $\{\tilde{z} = 0, w = 0\}$, the Levi-nondegenerate direction by $\{z_n = 0, w = 0\}$, the Levi form at 0 by

$$\langle \tilde{z}, \bar{\tilde{z}} \rangle := \sum_{j=1}^{n-1} \varepsilon^j |z_j|^2, \quad \varepsilon^j = 1 \text{ for } 0 \leq j \leq r, \quad \varepsilon^j = -1 \text{ for } r+1 \leq j \leq n-1, \quad (2.3)$$

and Ebenfelt's tensor by $(z_n)^2 \bar{z}_n$. The canonical Hermitian form (i.e., the Levi form) and the canonical cubic form, associated with M , enable us to define the differential operator

$$\mathbf{tr} := \sum_{j=1}^{n-1} \varepsilon^j \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$$

on the space of formal series $\Phi(\tilde{z}, \bar{\tilde{z}}, u)$. We have the identity

$$\mathbf{tr} \langle \tilde{z}, \bar{\tilde{z}} \rangle = n - 1.$$

We are now in the position to proceed with the normalization procedure. As presence of the Levi-nondegenerate direction for a hypersurface in \mathbb{C}^{n+1} , $n \geq 2$ makes the normalization procedure significantly different from the one in the two-dimensional case, we consider the two-dimensional case separately in Section 3, and then consider the high-dimensional case in Section 4.

3. THE TWO-DIMENSIONAL CASE

Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface with generic Levi degeneracy at the point $0 \in M$. Choosing local holomorphic coordinates $(z, w) = (z, u + iv)$ near the origin as described before, we represent $(M, 0)$ by a defining equation

$$v = \Phi(z, \bar{z}, u) = \sum_{k,l \geq 0} \Phi_{kl}(u) z^k \bar{z}^l$$

with $\Phi(0) = 0$, $d\Phi(0) = 0$, $\Phi_{11}(0) = 0$, $\Phi_{21}(0) = 1$. We perform a transformation $w \mapsto w + Q(z)$, where $Q(z)$ is a cubic polynomial, in order to eliminate the pure quadratic and pure cubic terms in Φ . Hence we end up with a hypersurface

$$v = \Phi(z, \bar{z}, u) = \sum_{k,l \geq 0} \Phi_{kl}(u) z^k \bar{z}^l, \quad (3.1)$$

$$\Phi(0) = \Phi_{10}(0) = \Phi_{11}(0) = \Phi_{20}(0) = \Phi_{30}(0) = 0, \quad \Phi_{21}(0) = 1.$$

In what follows we consider only transformations, preserving the form (3.1). The Levi-degeneracy set $\Sigma \subset M$ is a totally real manifold

$$\Sigma = \{ \operatorname{Re} z = \chi(\operatorname{Im} z, u), \quad \operatorname{Im} w = \tau(\operatorname{Im} z, u) \}, \quad \chi(0) = \tau(0) = 0, \quad d\chi(0) = d\tau(0) = 0. \quad (3.2)$$

3.1. Formal normalization in the two-dimensional case.

We start with the proof of the formal version of Theorem 1. Let us introduce weights for the variables (z, w) in the following way:

$$[z] = 1, [w] = 3.$$

These weights correspond to the choice of a *model* for hypersurfaces of the form (3.1), given by

$$v = 2\operatorname{Re}(z^2\bar{z}). \quad (3.3)$$

This model, first introduced in [2] and later used in [11], is weighted homogeneous with respect to this choice of weights. Then for any formal power series $f(z, w) \in \mathbb{C}[[z, w]]$ without constant terms we get the formal expansion

$$f(z, w) = \sum_{m \geq 1} f_m(z, w),$$

where each $f_m(z, w)$ is a weighted homogeneous polynomial of weight m . Similarly, for the right-hand side we get the expansion

$$\Phi = 2\operatorname{Re}(z^2\bar{z}) + \sum_{m \geq 4} \Phi_m(z, \bar{z}, u)$$

(so that any hypersurface (3.1) can be interpreted as a perturbation of the model (3.3)).

If now $M = \{v = \Phi(z, \bar{z}, u)\}$ and $M^* = \{v^* = \Phi^*(z^*, \bar{z}^*, u^*)\}$ are two hypersurfaces, satisfying (3.1), and $z^* = f(z, w)$, $w^* = g(z, w)$ is a formal invertible transformation, transforming $(M, 0)$ into $(M^*, 0)$, we obtain the identity

$$\operatorname{Im} g(z, w)|_{w=u+i\Phi(z, \bar{z}, u)} = \Phi^*(f(z, w), \overline{f(z, w)}, \operatorname{Re} g(z, w))|_{w=u+i\Phi(z, \bar{z}, u)}. \quad (3.4)$$

Collecting in (3.4) terms of the low weights 1, 2, we get $g_1 = g_2 = 0$. For the weight 3 we obtain $f_1 = \lambda z$, $g_3 = \lambda^3 w$, $\lambda = f_z(0, 0) \in \mathbb{R} \setminus \{0\}$. Thus the initial transformation $F = (f, g)$ can be uniquely decomposed as $F = \tilde{F} \circ \Lambda$, where

$$\Lambda(z, w) = (\lambda z, \lambda^3 w), \lambda \in \mathbb{R} \setminus \{0\}, \quad (3.5)$$

is an automorphism of the model (3.3) and the weighted components of the new mapping \tilde{F} satisfy

$$f_1 = z, \quad g_1 = g_2 = 0, \quad g_3 = w. \quad (3.6)$$

Hence in what follows we can restrict ourself to transformations, satisfying (3.6) After these preparations we consider (3.4) as an infinite series of weighted homogeneous equations, which can be written for any fixed weight $m \geq 4$ as

$$\operatorname{Re}(ig_m + (2z\bar{z} + \bar{z}^2)f_{m-2})|_{w=u+i(z^2\bar{z}+z\bar{z}^2)} = \Phi_m^* - \Phi_m + \dots, \quad (3.7)$$

where dots stands for a polynomial in z, \bar{z}, u and f_{j-2}, g_j with $j < m$ and their derivatives in u (here $f_{j-2} = f_{j-2}(z, u)$, $g_j = g_j(z, u)$, $\Phi_j = \Phi_j(z, \bar{z}, u)$).

Let us denote by \mathcal{F} the space of formal real-valued power series $\Phi(z, \bar{z}, u) = \sum_{m \geq 4} \Phi_m(z, \bar{z}, u)$, satisfying (3.1), by $\mathcal{N} \subset \mathcal{F}$ the subspace of series satisfying the normalization conditions (1.1), and by \mathcal{G} the space of pairs (f, g) of formal power series without constant term, satisfying (3.6). In view of (3.7), in order to prove the formal version of Theorem 1, it is sufficient now to prove

Proposition 3.1. *For the linear operator*

$$L(f, g) := \operatorname{Re} (ig + (2z\bar{z} + \bar{z}^2)f) \big|_{w=u+i(z^2\bar{z}+z\bar{z}^2)}$$

we have the direct sum decomposition $\mathcal{F} = L(\mathcal{G}) \oplus \mathcal{N}$.

Indeed, it follows from Proposition 2.1 that if $m \geq 4$ is an integer and all f_{j-2}, g_j, Φ_j^* with $j < m$ are already determined, then one can uniquely choose the collection (f_{m-2}, g_m, Φ_m^*) in such a way that (3.7) is satisfied and $\Phi_m^* \in \mathcal{N}$. This implies the existence and uniqueness of the desired normalized mapping (f, g) and the normalized right-hand side $\Phi^*(z, \bar{z}, u)$.

Proof of Proposition 2.1. The statement of the proposition is equivalent to the fact that an equation $L(f, g) = \Psi(z, \bar{z}, u)$, $(f, g) \in \mathcal{G}$ in (f, g) has a unique solution, modulo \mathcal{N} in the right-hand side, for any fixed $\Psi \in \mathcal{F}$. To simplify the calculations, we replace an equation $L(f, g) = \Psi(z, \bar{z}, u)$ by the equation

$$2L(f, g) = \Psi(z, \bar{z}, u), \quad (f, g) \in \mathcal{G}, \Psi \in \mathcal{F}, \quad (3.8)$$

which we solve in f, g . We use expansions of the form

$$f(z, u + i(z^2\bar{z} + z\bar{z}^2)) = f(z, u) + f_u(z, u)i(z^2\bar{z} + z\bar{z}^2) + \frac{1}{2}f_{uu}(z, u)i^2(z^2\bar{z} + z\bar{z}^2)^2 + \dots$$

Substituting into (3.8) we get the equation

$$\begin{aligned} & i \left(g(z, u) + g_u(z, u)i(z^2\bar{z} + z\bar{z}^2) + \frac{1}{2}g_{uu}(z, u)i^2(z^2\bar{z} + z\bar{z}^2)^2 + \dots \right) + \\ & + (2z\bar{z} + \bar{z}^2) \left(f(z, u) + f_u(z, u)i(z^2\bar{z} + z\bar{z}^2) + \frac{1}{2}f_{uu}(z, u)i^2(z^2\bar{z} + z\bar{z}^2)^2 + \dots \right) + \\ & + \{\text{complex conjugate terms}\} = \Psi(z, \bar{z}, u). \end{aligned} \quad (3.9)$$

We expand $f(z, w)$ as $f = \sum_{k \geq 0} f_k(w)z^k$, and similarly for g . Then, collecting in (3.9) terms of bi-degree $(k, 0)$, $k \geq 3$ in z, \bar{z} , we get

$$ig_k(u) = \Psi_{k0}(u). \quad (3.10)$$

Gathering then terms of bi-degree $(k+1, 1)$ with $k \geq 4$, we get

$$-g'_{k+1}(u) + 2f_k(u) = \Psi_{k+1,0}(u). \quad (3.11)$$

The equations (3.10), (3.11) enable us to determine g_k with $k \geq 3$ and f_k with $k \geq 4$ uniquely. We then proceed further with comparing terms of fixed bi-degree. Gathering terms of bi-degrees $(1, 0)$ and $(3, 1)$ respectively, we get

$$\begin{aligned} ig_1(u) &= \Psi_{10}(u) \\ -g'_1(u) + 2f_2(u) &= \Psi_{31}(u). \end{aligned} \quad (3.12)$$

The system (3.12) determines the pair (g_1, f_2) uniquely. Further, gathering $(1, 1)$ terms, we get

$$4\operatorname{Re} f_0(u) = \Phi_{11}(u),$$

so that only $\operatorname{Im} f_0(u)$ needs to be determined. Gathering then terms of bi-degrees $(2, 0)$, $(4, 1)$ and $(3, 2)$ respectively, and separating real and imaginary parts, we obtain the system

$$\begin{aligned} -\operatorname{Im} g_2(u) &= \operatorname{Re} \Psi_{20}(u) \\ \operatorname{Re} g_2(u) - \operatorname{Im} f_0(u) &= \operatorname{Im} \Psi_{20}(u) \\ -\operatorname{Re} g_2'(u) + 2\operatorname{Re} f_3(u) - \operatorname{Im} f_0'(u) &= \operatorname{Re} \Psi_{41}(u) \\ -\operatorname{Im} g_2'(u) + 2\operatorname{Im} f_3(u) &= \operatorname{Im} \Psi_{41}(u) \\ -\operatorname{Re} g_2'(u) + \operatorname{Re} f_3(u) - 3\operatorname{Im} f_0'(u) &= \operatorname{Re} \Psi_{32}(u). \end{aligned} \tag{3.13}$$

The first and the fourth equations in (3.13) determine the unknowns $\operatorname{Im} g_2, \operatorname{Im} f_3$ uniquely. Differentiating the second equation and considering it together with the third and the fifth equations, we get a real linear system for $\operatorname{Re} g_2', \operatorname{Re} f_3, \operatorname{Im} f_0'$ with a non-zero determinant, thus $\operatorname{Re} g_2', \operatorname{Re} f_3, \operatorname{Im} f_0'$ are also determined uniquely. Since $f(0, 0) = 0$, we have $f_0(0) = 0$, and since Ψ does not contain terms of weight 2, we have $\Psi_2(0) = 0$, so that from the second equation we obtain $\operatorname{Re} g_2(0) = 0$, and this determines $\operatorname{Re} g_2, \operatorname{Re} f_3, \operatorname{Im} f_0$ uniquely.

It remains to determine g_0 and f_1 . Gathering terms of bi-degrees $(0, 0)$, $(2, 1)$ and $(4, 2)$, and separating real and imaginary parts, we obtain the system

$$\begin{aligned} -2\operatorname{Im} g_0(u) &= \Psi_{00}(u) \\ -2\operatorname{Re} g_0'(u) + 3\operatorname{Re} f_1(u) &= \operatorname{Re} \Psi_{21}(u) \\ \operatorname{Im} f_1(u) &= \operatorname{Im} \Psi_{21}(u) \\ -\operatorname{Im} g_3'(u) + \operatorname{Im} g_0''(u) + \operatorname{Re} f_1'(u) + \operatorname{Im} f_4(u) &= \operatorname{Im} \Psi_{42}(u). \end{aligned} \tag{3.14}$$

We then determine, respectively, $\operatorname{Im} g_0$ from the first, $\operatorname{Im} f_1$ from the third, $\operatorname{Re} f_1'$ from the fourth, and $\operatorname{Re} g_0'$ from the second equations in (3.14). Since we have $f_1(0) = g_0'(0) = 1$, we determine from here $\operatorname{Re} f_1, \operatorname{Re} g_0$ uniquely (where f_4 is already determined from (3.11)).

Thus the map (f, g) is uniquely determined. Since the collection of all $\operatorname{Re} \Psi_{kl}, \operatorname{Im} \Psi_{kl}$ considered above must vanish for $\Psi \in \mathcal{N}$, while all the remaining $\operatorname{Re} \Psi_{kl}, \operatorname{Im} \Psi_{kl}$ for $\Psi \in \mathcal{N}$ can be arbitrary, this proves the proposition. \square

The formal version of Theorem 1 is proved now. Using the fact that any transformation (3.5) preserves the normalization conditions (1.1), we can also formulate

Corollary 3.2 (see also [2],[11]). *The group of formal invertible transformations, preserving the germ at 0 of the model hypersurface (3.3), consists of the dilations (3.5).*

Corollary 3.3. *If $(N, 0)$ and $(\tilde{N}, 0)$ are two different normal forms of a fixed germ (M, p) with generic Levi degeneracy at p , then there exists a linear transformation Λ , as in (3.5), which maps $(N, 0)$ into $(\tilde{N}, 0)$.*

3.2. Convergence of the normalizing transformation.

Our next goal is to prove that the constructed normalizing transformation $F = (f, g)$, satisfying (3.6), is in fact convergent. We do that by presenting F as a composition of certain holomorphic transformations. Each of the transformation has a clear geometric interpretation, that we address below. For the set-up of the theory of Segre varieties see, e.g., [1].

Canonical pair of foliations in the Levi degeneracy set. Let Σ be the Levi degeneracy set of a real-analytic hypersurface $M \subset \mathbb{C}^2$ with generic Levi degeneracy at the point $0 \in \Sigma \subset M$. We

first define the following slope (line) field in the totally real submanifold Σ . Choose a point $p \in \Sigma$ and coordinates (z, w) vanishing at p , where M takes the form (3.1). Clearly these coordinates can be chosen polynomial with coefficients depending analytically on p .

Let N denotes a (formal) normal form (1.1) of M at p , F a formal transformation, mapping (M, p) onto $(N, 0)$, and $e := (0, 1) \in \mathbb{C}^2$. We then define a slope at p as follows:

$$l(p) := \text{span}_{\mathbb{R}} \{(dF|_p)^{-1}(e)\} \subset T_p M.$$

It follows from Corollary 3.3 that the definition of l is independent of the choice of normal form. Moreover, the desired slope can be also defined without using formal transformations. Indeed, it follows from the normal form construction that, as soon as the initial weighted polynomials $\{\Phi_j^*, f_{j-2}, g_j, 4 \leq j \leq m\}$ for some $m \geq 4$ have been determined, they do not change after further normalization of terms of higher weight. Hence, solving the equations (3.7) for m sufficiently large, we uniquely determine $dF|_p$. It is not difficult to see that the constructed slope field is analytic (i.e., depends analytically on a point $p \in \Sigma$). Indeed, the explicit construction in the beginning of the section shows that each fixed weighted polynomial Φ_m , as in (3.1), depends on p analytically (this can be verified from the parameter version of the implicit function theorem). Hence polynomials f_m and g_m depends on p analytically, as it is obtained by solving a system of linear equations with fixed nondegenerate matrix in the left-hand and right-hand side analytic in p (the latter fact can be seen from the proof of Proposition 2.1). We immediately conclude that $dF|_p$ depends on p analytically, and so does $l(p)$.

We then integrate the analytic slope field $l(p)$ and obtain a canonical (non-singular) real-analytic foliation \mathcal{T} of the totally real manifold Σ of Levi-degenerate points.

Definition 3.4. We call each of the leaves of the foliation \mathcal{T} a *degenerate chain*.

Each degenerate chain $\gamma \subset \Sigma$ at each point $p \in \gamma$ is transverse to the complex tangent $T_p^{\mathbb{C}} M$. The second canonical foliation in Σ corresponds to the slope field

$$c(p) := T_p^{\mathbb{C}} M \cap T_p \Sigma.$$

Integrating $c(p)$ we obtain another canonical foliation \mathcal{S} in Σ , which is everywhere tangent to $T^{\mathbb{C}} M$. Both foliations \mathcal{T} and \mathcal{S} are transverse to each other and are biholomorphic invariants of $(M, 0)$. We call them respectively *transverse* and *tangent* foliations.

Normalization of a chain and of the field of complex tangent vectors along the chain.

We start the construction of a convergent transformation, mapping a germ $(M, 0)$, as in (3.1), onto a germ $(N, 0)$, as in (1.1), by choosing the unique degenerate chain $\gamma \subset \Sigma$, passing through 0. Let us denote by $s(p)$ the leaf of the tangent foliation \mathcal{S} , passing through a point $p \in \Sigma$. As Σ is totally real in \mathbb{C}^2 , we may perform a local holomorphic coordinate change near 0 in such a way that the form (3.1) is preserved, the Levi degeneracy set Σ is transformed into

$$\Pi = \{\text{Re } z = 0, \text{Im } w = 0\}, \quad (3.15)$$

the degenerate chain γ into

$$\Gamma = \{z = 0, v = 0\} \subset \Pi, \quad (3.16)$$

and each leaf s_p , $p \in \gamma$ of the tangent foliation near 0 into $\{\text{Re } z = 0, w = b\}$, $b \in \mathbb{R}$. Thus for the transformed hypersurface M^* , all complex tangent spaces $T_p^{\mathbb{C}} M$, $p \in \Gamma$, are of the form $\{w = 0\}$. Since Σ is a plane for M^* , we have

$$\Phi_{11}^*(u) = 0,$$

expressing the fact that the Levi form of M vanishes along Γ . Furthermore, since M^* contains Γ , we also obtain the condition $\Phi_{00}^*(u) = 0$. In what follows we consider only transformations,

preserving Γ (we will prove later that when M is already in the normal form, then the curve Γ coincides with the degenerate chain, passing through 0). Note that in the normal form (1.1) the Levi degeneracy set Σ is *not necessarily flat*, as in (3.15), and neither are the leaves s_p , $p \in \Gamma$. However, all the complex tangents $T_p^{\mathbb{C}} M$, $p \in \Gamma$, remain $\{w = 0\}$, and the condition $\Phi_{11}^*(u) = 0$ is also preserved.

Normalization of Segre varieties along a chain. The next step in the normalization procedure is the elimination of $(k, 0)$ terms in the expansion of Φ , which is sometimes addressed as transfer to *normal coordinates* (see [1]). Geometrically, this step can be interpreted as straightening of the Segre varieties Q_p , $p \in \Gamma$. According to [5], [1], we perform the unique transformation of the form

$$z^* = z, \quad w^* = w + g(z, w), \quad g(0, w) = 0, \quad (3.17)$$

which maps M into a hypersurface with $\Phi_{k0}^* = 0$, $k \geq 0$. This transformation preserves the curve Γ . One needs only to take control of the fact that the conditions $\Phi_{11}(u) = 0$ and $\Phi_{21}(0) = 1$ are preserved. Indeed, one clearly has $\Phi_{11}^*(u^*) = 0$, since the Levi form of M^* vanishes along Γ . The substitution $z^* = z$, $w^* = u + i\Phi(z, \bar{z}, u) + O(|z|)$ into $v^* = \Phi^*(z^*, \bar{z}^*, u^*)$ also shows $\Phi_{21}^*(0) = 1$.

Normalization of the Segre map. This step can be interpreted as a normalization of the Segre map. The latter one, since Segre varieties are determined by their 2-jets, can be regarded as a map $p \mapsto j^2 Q_p|_{z=0}$ of $(\mathbb{C}^2, 0)$ into the 2-jet space of Segre varieties at the point with $z = 0$. We normalize the Segre map in this step by the condition

$$p = (\xi, \eta) \mapsto (\bar{\eta}, 2i\xi^2, 4i\bar{\xi} + O(\bar{\xi}^2)). \quad (3.18)$$

Let us introduce the subspace $\mathcal{D} \subset \mathcal{F}$, which consists of all convergent power series of the form

$$\sum_{k, l \geq 2} \Psi_{kl}(u) z^k \bar{z}^l.$$

Our goal is to bring a hypersurface M , obtained in the previous step, to such a form that all terms of weight ≥ 4 in Φ belong to the space \mathcal{D} , i.e., to bring the defining equation to the form

$$v = P(z, \bar{z}) \mod \mathcal{D}. \quad (3.19)$$

Thus the subspace of terms to be removed from Φ in this step is transverse to \mathcal{D} , and adding to Φ an element of \mathcal{D} does not change the desired form of it, that is why it is convenient for us to use identities modulo \mathcal{D} .

Consider a hypersurface M , obtained in the previous step. We first make Φ_{21} independent of u . Let $\lambda(u)$ be an analytic function with $\lambda(0) = 1$ such that $\lambda^2(u)\bar{\lambda}(u) = \Phi_{21}(u)$. We perform the biholomorphic change

$$z^* = z\lambda(w), \quad w^* = w.$$

Using expansions of the form $h(u + iv) = h(u) + ih'(u)v + \dots$, we compute

$$\Phi(z, \bar{z}, u) = \Phi^*(z^*, \bar{z}^*, u^*) = \Phi^*(z\lambda(u), \bar{z}\bar{\lambda}(u), u) \mod \mathcal{D},$$

provided $(z, w) \in M$ (recall that all $(k, 0)$ -terms are removed from Φ). Thus

$$\Phi_{21}(u) = \Phi_{21}^*(u)\lambda^2(u)\bar{\lambda}(u),$$

so that M is mapped into a hypersurface of the form

$$v = 2\operatorname{Re}(z^2 \bar{z}) + \sum_{k, l \geq 1, k+l \geq 4} \Phi_{kl}^*(u) z^k \bar{z}^l,$$

as required. Clearly, Γ is preserved and $\Phi_{11}^* = 0$.

Second, for the hypersurface M , obtained in the previous step, we remove all $(k, 1)$ -terms in Φ with $k \geq 3$ by a transformation

$$z^* = z + f(z, w), \quad w^* = w, \quad f(z, w) = O(|z|^2).$$

We have

$$\Phi^*(z + f(z, u + i\Phi(z, \bar{z}, u)), \bar{z} + \overline{f(z, u + i\Phi(z, \bar{z}, u))}, u) = \Phi(z, \bar{z}, u). \quad (3.20)$$

Furthermore $\Phi_{21}^*(u) = \Phi_{21}(u)$ and $\Phi_{11}^*(u) = \Phi_{11}(u) = 0$.

We expand the defining function Φ as

$$\Phi(z, \bar{z}, u) = 2\operatorname{Re} (z^2 \bar{z} + z^2 \phi(z, u) \bar{z}) \mod \mathcal{D}$$

for an appropriate $\phi(z, u) = O(|z|)$. Put $f =: zh$. We compute

$$(z + f)^2 (\bar{z} + \bar{f}) = z^2 \bar{z} (1 + 2h + h^2) \mod \mathcal{D}.$$

We then determine $h(z, u), f(z, u)$ from the functional equation

$$2h + h^2 = \phi(z, u), \quad (3.21)$$

expressing the condition that M is mapped into a hypersurface M^* given by (3.19). Now suitable h can be obtained by the implicit function theorem. Note that the required transformation, removing the $(k, 1)$ terms, is unique. It is straightforward that the Segre map is given by (3.18).

Fixing a parametrization for a chain. We claim now that $\operatorname{Re} \Phi_{32}(u) = 0$ in (3.19). Indeed, consider the (formal) transformation $F = (f, g)$, bringing a hypersurface (3.19) into normal form, and study the equation (3.4), applied to it. Collecting terms with $z\bar{z}u^l$, $l \geq 0$, we first obtain $\operatorname{Re} f_0(u) = 0$. Next, gathering terms with $z^2 \bar{z}^0 u^1$, $z^4 \bar{z}^1 u^0$, $z^3 \bar{z}^2 u^0$ and separating the real and imaginary parts, it is straightforward to check that we obtain precisely the equations (3.13) (and also the differentiated second equation in (3.13)), evaluated at $u = 0$, with $\Psi_{20}(0) = \Psi'_{20}(0) = \Psi_{41}(0) = 0$ and $\operatorname{Re} \Psi_{32}(0) = 2\operatorname{Re} \Phi_{32}(0)$ (this follows from the partial normalization (3.19) of M and the normalization conditions (1.1) for the target M^*). Moreover, $\operatorname{Im} f_0(0) = \operatorname{Im} f'_0(0) = 0$ thanks to the fact that Γ is a degenerate chain. We immediately conclude from (3.13) that $\operatorname{Re} g_2(0) = \operatorname{Re} g'_2(0) = \operatorname{Im} g'_2(0) = \operatorname{Re} f_3(0) = \operatorname{Im} f_3(0) = 0$ and $\operatorname{Re} \Phi_{32}(0) = 0$. Since the prenormal form (3.19) is invariant under the real shifts $w \mapsto w + u_0$, $u_0 \in \mathbb{R}$, and $\Gamma \subset M$ is a degenerate chain, we similarly conclude that $\operatorname{Re} \Phi_{32}(u_0) = 0$ for any small $u_0 \in \mathbb{R}$ in (3.19), as required.

It remains to achieve the last normalization condition $\operatorname{Im} \Phi_{42}(u) = 0$, using an appropriate *gauge* transformation

$$z \mapsto f(w)z, \quad w \mapsto g(w), \quad f(0) \neq 0, g'(0) \neq 0. \quad (3.22)$$

We do so by choosing a gauge transformation

$$z^* = z(q'(w))^{1/3}, \quad w^* = q(w)$$

for an appropriate $q(w)$ with $\operatorname{Im} q(u) = 0$, $q(0) = 0$, $q'(0) \neq 0$, which can be interpreted as a choice of parametrization for the degenerate chain γ , determined in the previous step. We apply the above transformation to a hypersurface, satisfying (3.19) and $\operatorname{Re} \Phi_{32}(u) = 0$. Recall that \mathcal{N} denotes the space of power series in z, \bar{z}, u of weight ≥ 4 , satisfying the normalization conditions

(1.1). Then, using expansions of kind $h(u + iv) = h(u) + ih'(u)v + \dots$, we compute:

$$\begin{aligned} v^* &= q'(u)v \mod \mathcal{N}, \\ z^{*2}\bar{z}^* &= z^2\bar{z}q'(u) \left(1 + i\frac{q''(u)}{q'(u)}v\right)^{2/3} \left(1 - i\frac{q''(u)}{q'(u)}v\right)^{1/3} \mod \mathcal{N} = \\ &= q'(u)z^2\bar{z} + \frac{i}{3}q''(u)z^4\bar{z}^2 \mod \mathcal{N}, \\ v^* - 2\operatorname{Re}(z^{*2}\bar{z}^*) &= q'(u)(v - 2\operatorname{Re}(z^2\bar{z})) + \frac{i}{3}q''(u)(z^4\bar{z}^2 - z^2\bar{z}^4) \mod \mathcal{N}. \end{aligned}$$

We conclude that $\operatorname{Im} \Phi_{42}^* = q'\operatorname{Im} \Phi_{42} + \frac{1}{3}q''$, so that the condition $\operatorname{Im} \Phi_{42}^*(u) = 0$ leads to a second order *nonsingular* ODE. Solving it with some initial condition $q'(0) \neq 0$, we finally obtain a hypersurface of the form (3.19), satisfying $\operatorname{Re} \Phi_{32}(u) = \operatorname{Im} \Phi_{42}(u) = 0$, as required for the complete normal form. It is not difficult to see, performing similar calculations, that the gauge transformation chosen to achieve $\operatorname{Im} \Phi_{42} = 0$ must have the above form $z^* = z(q'(w))^{1/3}$, $w^* = q(w)$ and hence is unique up to the choice of the real parameter $q'(0)$, corresponding to the action of the group (3.5). Thus, remarkably,

we can canonically, up to the action of the group of dilations (3.5), choose a parametrization on each degenerate chain.

Theorem 1 is completely proved now. The proof shows also that in the normal form (1.1) the unique degenerate chain, passing through the origin, is given by (3.16). In addition, we can see that

a transformation, bringing a germ of a real-analytic hypersurface $M \subset \mathbb{C}^2$ with generic Levi degeneracy at 0 into normal form, is completely determined by fixing a tangent vector at 0 to the degenerate chain, passing through 0.

4. THE HIGHER DIMENSIONAL CASE

Let $M \subset \mathbb{C}^{n+1}$, $n \geq 2$ be a real-analytic hypersurface with generic Levi degeneracy at 0. Choosing local coordinates for M as discussed in Section 2, and performing an additional polynomial transformation removing harmonic terms of degrees 2 and 3, we can present M locally near the origin as

$$v = \langle \tilde{z}, \bar{\tilde{z}} \rangle + 2\operatorname{Re}(z_n^2 \bar{z}_n) + O(|\tilde{z}|^2|z_n| + |z|^4 + u|z| + u^2). \quad (4.1)$$

(compare with the initial simplification in [6]).

4.1. Formal normalization in the higher dimensional case. A crucial step in the construction of a normal form is again a good choice of weights for a hypersurface given by (4.1). We introduce weights as follows:

$$[\tilde{z}] := 3, \quad [z_n] := 2, \quad [w] := 6.$$

Then the hypersurface

$$v = P(z, \bar{z}), \quad (4.2)$$

where

$$P(z, \bar{z}) := \langle \tilde{z}, \bar{\tilde{z}} \rangle + 2\operatorname{Re}(z_n^2 \bar{z}_n) = \sum_{j=1}^{n-1} \varepsilon^j |z_j|^2 + 2\operatorname{Re}(z_n^2 \bar{z}_n)$$

is a weighted homogeneous polynomial of weight 6, becomes a "model" for the class of hypersurfaces (4.1) that can be written as

$$v = P(z, \bar{z}) + \sum_{m \geq 7} \Phi_m(z, \bar{z}, u), \quad (4.3)$$

where each $\Phi_m(z, \bar{z}, u)$ is homogeneous of weight m .

Let now $M = \{v = \Phi(z, \bar{z}, u)\}$ and $M^* = \{v^* = \Phi^*(z^*, \bar{z}^*, u^*)\}$ be two hypersurfaces in \mathbb{C}^{n+1} , satisfying (4.1), and $z^* = f(z, w)$, $w^* = g(z, w)$ a formal invertible transformation, transforming $(M, 0)$ into $(M^*, 0)$. We have the identity

$$\operatorname{Im} g(z, w)|_{w=u+i\Phi(z, \bar{z}, u)} = \Phi^*(f(z, w), \overline{f(z, w)}, \operatorname{Re} g(z, w))|_{w=u+i\Phi(z, \bar{z}, u)}. \quad (4.4)$$

For a mapping $(f, g) = (f^1, \dots, f^n, g) : (\mathbb{C}^{n+1}, 0) \mapsto (\mathbb{C}^{n+1}, 0)$ we group the first $n-1$ components in the vector function

$$\check{f} = (f^1, \dots, f^{n-1}).$$

We decompose the mapping into a sum of weighted homogeneous polynomials as

$$f = \sum_{m \geq 2} f_m(z, w), \quad \check{f} = \sum_{m \geq 2} \check{f}_m(z, w), \quad g = \sum_{m \geq 2} g_m(z, w)$$

and consider terms of a fixed weight in (4.4). Collecting terms of the low weights 2, ..., 5, we get in view of (4.1),

$$\check{f}_2 = g_2 = \dots = g_5 = 0.$$

For the weight 6 we have

$$\check{f}_3 = \lambda^3 C \check{z}, \quad f_2^n = \rho \lambda^2 z_n, \quad g_6 = \rho \lambda^6 w,$$

where

$$\langle C \check{z}, \overline{C \check{z}} \rangle = \rho \langle \check{z}, \bar{\check{z}} \rangle, \quad \lambda > 0, \quad C \in \operatorname{GL}(n-1, \mathbb{C}), \quad \rho \in \{1, -1\}$$

(in fact, for $r = n-1$ we have $\rho = 1$ only). Thus the initial transformation $F = (f, g)$ can be uniquely decomposed as $F = \tilde{F} \circ \Lambda$, where

$$\begin{aligned} \Lambda(z, w) &= (\lambda^3 C \check{z}, \rho \lambda^2 z_n, \rho \lambda^6 w), \\ \langle C \check{z}, \overline{C \check{z}} \rangle &= \rho \langle \check{z}, \bar{\check{z}} \rangle, \quad C \in \operatorname{GL}(n-1, \mathbb{C}), \quad \lambda > 0, \quad \rho \in \{1, -1\} \end{aligned} \quad (4.5)$$

is an automorphism of the model (4.2), and the weighted components of \tilde{F} satisfy

$$\check{f}_2 = 0, \quad \check{f}_3 = \check{z}, \quad f_2^n = z_n, \quad g_2 = \dots = g_5 = 0, \quad g_6 = w. \quad (4.6)$$

Thus in what follows we only consider maps, satisfying (4.6). Proceeding further with weighted identities arising from (4.4), and using (4.3), we obtain, similarly to the two-dimensional case

$$\operatorname{Re} \left(i g_m + \langle \check{f}_{m-3}, \bar{\check{z}} \rangle + \left(2 z_n \bar{z}_n + (\bar{z}_n)^2 \right) f_{m-4}^n \right) |_{w=u+iP(z, \bar{z})} = \Phi_m^* - \Phi_m + \dots, \quad (4.7)$$

for any fixed weigh $m \geq 7$, where dots stand for a polynomial in z, \bar{z}, u and $\check{f}_{j-3}(z, w), f_{j-4}^n(z, w), g_j(z, w)$ with $j < m$, with $w = u + iP(z, \bar{z})$.

We now decompose the space \mathcal{F}^n of formal real-valued power series

$$\Phi(z, \bar{z}, u) = \sum_{m \geq 7} \Phi_m(z, \bar{z}, u),$$

satisfying (4.1), into the direct some of the image \mathcal{R}^n of the operator on left-hand side (4.7), called the *Chern-Moser operator* associated to P :

$$L^n(f, g) := \operatorname{Re}(ig + P_z f)|_{w=u+iP(z, \bar{z})} = \operatorname{Re}\left(ig + \langle \check{f}, \bar{\check{z}} \rangle + \left(2z_n \bar{z}_n + (\bar{z}_n)^2\right)f^n\right)|_{w=u+iP(z, \bar{z})},$$

acting on formal power series

$$(f, g) \in \mathbb{C}[[z, w]]^n \times \mathbb{C}[[z, w]],$$

and an appropriate normal subspace $\mathcal{N}^n \subset \mathcal{F}^n$. Let also \mathcal{G}^n denotes the space of all $(n+1)$ -tuples (f, g) of power series without constant terms, satisfying (4.6). As was already discussed in Section 3, in order to prove the formal version of Theorem 2, it remains to prove:

Proposition 4.1. *For the Chern-Moser operator L^n , we have the direct decomposition*

$$\mathcal{F}^n = L^n(\mathcal{G}^n) \oplus \mathcal{N}^n,$$

where $\mathcal{N}^n \subset \mathcal{F}^n$ is the subspace of series, satisfying the normalization conditions (1.3).

Proof. We have to prove that an equation

$$2L^n(f, g) = \Psi(z, \bar{z}, u), \quad (f, g) \in \mathcal{G}^n, \quad (4.8)$$

has a unique solution in (f, g) , modulo \mathcal{N}^n in the right-hand side, for any fixed $\Psi \in \mathcal{F}^n$. We expand (4.8) as

$$\begin{aligned} & i \left(g(z, u) + g_u(z, u) iP(z, \bar{z}) + \frac{1}{2} g_{uu}(z, u) i^2 (P(z, \bar{z}))^2 + \dots \right) \\ & + \left\langle \check{f}(z, u) + \check{f}_u(z, u) iP(z, \bar{z}) + \dots, \bar{\check{z}} \right\rangle + (2z_n \bar{z}_n + (\bar{z}_n)^2) \left(f^n(z, u) + f_u^n(z, u) iP(z, \bar{z}) + \dots \right) \\ & + \{\text{complex conjugate terms}\} = \Psi(z, \bar{z}, u), \end{aligned} \quad (4.9)$$

and each $f^s(z, u)$, $1 \leq s \leq n$, as

$$f^s = \sum f_{\alpha l}^s(w) \check{z}^\alpha z_n^l, \quad \check{z}^\alpha = z_1^{\alpha_1} \dots z_{n-1}^{\alpha_{n-1}}, \quad \alpha = (\alpha_1, \dots, \alpha_{n-1}) \in (\mathbb{Z}_{\geq 0})^{n-1}, \quad l \in \mathbb{Z}_{\geq 0},$$

and similarly for g . We expand the right-hand side as

$$\Psi(z, \bar{z}, u) = \sum \Psi_{k'kl'l}(\check{z}, \bar{\check{z}}, u) z_n^k \bar{z}_n^l, \quad k', k, l', l \in \mathbb{Z}_{\geq 0},$$

where each $\Psi_{k'kl'l}$ is bi-homogeneous of bi-degrees (k', l') in $(\check{z}, \bar{\check{z}})$ respectively. We are looking for the normal space \mathcal{N}^n of the form

$$\mathcal{N}^n = \bigoplus \mathcal{N}_{k'kl'l}, \quad \mathcal{N}_{k'kl'l} \subset \{ \Psi(z, \bar{z}, u) = \Psi_{k'kl'l}(\check{z}, \bar{\check{z}}, u) z_n^k \bar{z}_n^l \}.$$

We start by considering certain “mixed” terms in the basic equation (4.9) in order to simplify further calculations. First, we collect terms of the form $\mathbb{C}[[u]]z_n \bar{z}_n$ in (4.9) and get

$$4\operatorname{Re} f_{00}^n(u) = \Psi_{0101}(u). \quad (4.10)$$

Collecting then in (4.9) all terms of the form $\mathbb{C}[[u]]\check{z}^\alpha \bar{\check{z}}^\beta z_n \bar{z}_n$, $|\alpha| = |\beta| = 1$, and applying the \mathbf{tr} operator, we obtain

$$-4\operatorname{Im} (f_{00}^n)'(u) = \frac{1}{n-1} \mathbf{tr} \Psi_{1111}. \quad (4.11)$$

Thus $\operatorname{Im} (f_{00}^n)'(u)$ is determined uniquely. In view of $f_{00}^n(0) = 0$, there exists unique $f_{00}^n(u)$ satisfying (4.10) and (4.11). Consequently we can set

$$\mathcal{N}_{0101} = 0, \quad \mathcal{N}_{1111} = \ker \mathbf{tr}, \quad (4.12)$$

for the components of the normal space \mathcal{N}^n .

Second, we collect in (4.9) all the terms $\mathbb{C}[u]\check{z}^\alpha, |\alpha| = 1$, as well as terms of the form $\mathbb{C}[u]\check{z}^\alpha z_n \bar{z}_n^2, |\alpha| = 1$, and obtain the system

$$\begin{aligned} i \sum_{|\alpha|=1} g_{\alpha 0}(u) \check{z}^\alpha + \langle \check{z}, \overline{\check{f}_{00}}(u) \rangle &= \Psi_{1000}, \\ - \sum_{|\alpha|=1} g'_{\alpha 0}(u) \check{z}^\alpha - i \langle \check{z}, \overline{\check{f}'_{00}}(u) \rangle &= \Psi_{1102}. \end{aligned} \quad (4.13)$$

As the form $\langle \cdot, \cdot \rangle$ is nondegenerate, (4.13) enables us to determine $\check{f}'_{00}(u), g'_{\alpha 0}(u), |\alpha| = 1$, uniquely. Since we have $\check{f}_{00}(0) = 0$, and also $g_{\alpha 0}(0) = 0$ from (4.6), $|\alpha| = 1$, equations (4.13) completely determine $\check{f}_{00}(u), g_{\alpha 0}(u), |\alpha| = 1$. Thus we can set

$$\mathcal{N}_{1000} = \mathcal{N}_{1102} = 0. \quad (4.14)$$

Having some of the initial terms of the mapping (f, g) determined, we proceed further with solving the equation (4.9). Collecting all terms of the form $\mathbb{C}[[z, u]]$, we get

$$ig(z, u) - ig_{00}(u) + \langle \check{z}, \overline{\check{f}_{00}}(u) \rangle + z_n^2 \overline{f_{00}^n}(u) = \Psi(z, 0, u). \quad (4.15)$$

Since $f_{00}^n(u)$ is already determined, the equation (4.15) enables us to determine all $g_{\alpha l}(u)$ uniquely, except $\text{Re } g_{00}$ (while $\text{Im } g_{00}$ is also uniquely determined). Keeping the latter conclusion in mind, we collect in (4.9) all terms of the form $\mathbb{C}[[z, u]]\check{z}^\alpha, |\alpha| = 1$, to obtain

$$\begin{aligned} -g_u(z, u) \langle \check{z}, \bar{z} \rangle + \langle \check{f}(z, u), \bar{z} \rangle - i \sum_{|\alpha|=1} \overline{g_{\alpha 0}}(u) \bar{z}^\alpha - \overline{g'_{00}}(u) \langle \check{z}, \bar{z} \rangle - i \sum_{|\alpha|=1} \overline{g_{\alpha 0}}(u) \bar{z}^\alpha + \\ + \sum_{|\alpha|=1} \langle \check{z}, \overline{\check{f}_{\alpha 0}}(u) \bar{z}^\alpha \rangle - i \langle \check{z}, \overline{\check{f}_{00}}(u) \rangle \langle \check{z}, \bar{z} \rangle + z_n^2 \sum_{|\alpha|=1} \overline{f_{\alpha 0}^n}(u) \bar{z}^\alpha - iz_n^2 \overline{(f_{00}^n)'}(u) \langle \check{z}, \bar{z} \rangle = \sum_{k', k \geq 0} \Psi_{k'k10} z_n^k, \end{aligned} \quad (4.16)$$

and also collect all terms of the form $\mathbb{C}[[z, u]]z_n \bar{z}_n$ to obtain

$$\begin{aligned} -g_u(z, u) z_n^2 \bar{z}_n + 2z_n \overline{f^n}(z, u) - i \overline{g_{01}}(u) \bar{z}_n - \overline{g'_{00}}(u) z_n^2 \bar{z}_n - i \langle \check{z}, \overline{\check{f}'_{00}}(u) \rangle z_n^2 \bar{z}_n \\ + 2 \overline{f_{00}^n}(u) z_n \bar{z}_n + \overline{f_{01}^n}(u) z_n^2 \bar{z}_n - i \overline{(f_{00}^n)'}(u) z_n^4 \bar{z}_n = \sum_{k' \geq 0, k \geq 1} \Psi_{k'k01} z_n^k \bar{z}_n. \end{aligned} \quad (4.17)$$

Equations (4.16) and (4.17) enable us to determine all $f_{\alpha l}(u)$ uniquely, except $\check{f}_{\alpha 0}$ for $|\alpha| = 1$, f_{01}^n and f_{02} . To determine $f_{\alpha 0}^n$ with $|\alpha| = 1$ and \check{f}_{02} we first determine $f_{\alpha 0}^n$ by considering $\check{z}^\alpha z_n \bar{z}_n$ terms with $|\alpha| = 1$ in (4.17), and then \check{f}_{02} by considering $z_n^2 \bar{z}^\alpha$ terms with $|\alpha| = 1$ in (4.16). Thus we have determined uniquely all $f_{\alpha k}^n$ for all α, k and $\check{f}_{\alpha k}$ for $|\alpha| \neq 1$, and can set

$$\mathcal{N}_{k'k10} = \mathcal{N}_{k'k01} = 0 \quad \text{except } \mathcal{N}_{1010}, \mathcal{N}_{0201}, \mathcal{N}_{k'001}.$$

To determine the remaining terms in f, g , we consider in the equation (4.9) terms of the form $\mathbb{C}[[u]]\check{z}^\alpha \bar{z}^\beta z_n^2 \bar{z}_n, |\alpha| = |\beta| = 1$. This gives

$$\begin{aligned} 2\text{Im } g''_{00}(u) \langle \check{z}, \bar{z} \rangle + i \sum_{|\alpha|=1} \langle \check{f}'_{\alpha 0}(u) \check{z}^\alpha, \bar{z} \rangle - \\ - i \sum_{|\beta|=1} \langle \check{z}, \overline{\check{f}'_{\beta 0}}(u) \bar{z}^\beta \rangle + 2i(f_{01}^n)'(u) \langle \check{z}, \bar{z} \rangle - i \overline{(f_{01}^n)'}(u) \langle \check{z}, \bar{z} \rangle = \Psi_{1211}. \end{aligned} \quad (4.18)$$

For terms in $\mathbb{C}[[u]]\tilde{z}^\alpha\bar{\tilde{z}}^\beta$, $|\alpha| = |\beta| = 1$, we obtain from (4.16)

$$-2\operatorname{Re} g'_{00}(u)\langle\tilde{z}, \bar{\tilde{z}}\rangle + \sum_{|\alpha|=1} \langle\check{f}_{\alpha 0}(u)\tilde{z}^\alpha, \bar{\tilde{z}}\rangle + \sum_{|\beta|=1} \langle\check{z}, \overline{\check{f}_{\beta 0}(u)}\bar{\tilde{z}}^\beta\rangle = \Psi_{1010}. \quad (4.19)$$

Finally, for terms in $\mathbb{C}[[u]]z_n^2\bar{z}_n$ we obtain from (4.17)

$$-2\operatorname{Re} g'_{00}(u) + 2f'_{01}(u) + \overline{f'_{01}(u)} = \Psi_{0201}. \quad (4.20)$$

For a multiindex $\alpha \in (\mathbb{Z}_{\geq 0})^n$ with $|\alpha| = 1$, we use the notation $f_{\gamma 0}^\alpha$ for the unique component $f_{\gamma 0}^j$, $1 \leq j \leq n-1$, such that $\alpha_j = 1$. We first take the real part of (4.18), and then consider terms in $\mathbb{C}[[u]]\tilde{z}^\alpha\bar{\tilde{z}}^\beta$ with $|\alpha| = |\beta| = 1$, $\alpha \neq \beta$; second, we consider in (4.19) the terms in $\mathbb{C}[[u]]\tilde{z}^\alpha\bar{\tilde{z}}^\beta$ with $|\alpha| = |\beta| = 1$, $\alpha \neq \beta$. We get the system

$$i(f_{\alpha 0}^\beta)'(u) - i(\overline{f_{\beta 0}^\alpha})'(u) = (\operatorname{Re} \Psi_{1211})_{\tilde{z}^\alpha\bar{\tilde{z}}^\beta}, \quad f_{\alpha 0}^\beta(u) + \overline{f_{\beta 0}^\alpha}(u) = (\Psi_{\alpha 0 \beta 0})_{\tilde{z}^\alpha\bar{\tilde{z}}^\beta}. \quad (4.21)$$

Note that $\operatorname{Re} \Psi_{1211}$ is already a Hermitian form. Differentiating the second equation in (4.21), we obtain for each α, β as above a nondegenerate linear system for $(f_{\alpha 0}^\beta)'(u)$, $(\overline{f_{\beta 0}^\alpha})'(u)$, that we can solve uniquely. The compatibility of the solutions follows from the reality of Ψ . From the normalization conditions (4.6) for (f, g) we get $f_{\alpha 0}^\beta(0) = 0$ for $\alpha \neq \beta$, so that all $f_{\alpha 0}^\beta(0)$ with $\alpha \neq \beta$ can be determined uniquely. Next, applying \mathbf{tr} to (4.18) and considering the imaginary part, we get

$$\operatorname{Im} \left(2i(f_{01}^n)'(u) - i(\overline{f_{01}^n})'(u) \right) = \frac{1}{2(n-1)} \operatorname{Im} \left\{ \mathbf{tr} \Psi_{1211}(u) \right\}.$$

Differentiating (4.20) and considering the imaginary part, we obtain

$$\operatorname{Im} \left(2(f_{01}^n)'(u) + (\overline{f_{01}^n})'(u) \right) = \operatorname{Im} \Psi'_{0201}(u).$$

The two latter equations enable us to determine $(f_{01}^n)'(u)$ uniquely by setting

$$\mathcal{N}_{0201} = \ker \operatorname{Im}, \quad \mathcal{N}_{1211} = \ker \operatorname{Im} \mathbf{tr}.$$

From (4.6) we have $f_{01}^n(0) = 1$, so that $f_{01}^n(u)$ is completely determined. After that $\operatorname{Re} g'_{00}(u)$ is uniquely determined by considering the real part of (4.20), and setting (together with previous normalization)

$$\mathcal{N}_{0201} = 0.$$

and this uniquely determines $\operatorname{Re} g_{00}(u)$ thanks to the condition $g_{00}(0) = 0$ (recall that $\operatorname{Im} g_{00}(u)$ was already determined above). Finally, we determine $\operatorname{Re} f_{\alpha 0}^\alpha$ by considering terms in $\mathbb{C}[[u]]\tilde{z}^\alpha\bar{\tilde{z}}^\alpha$ in the real part of (4.19), and determine $\operatorname{Im} (f_{\alpha 0}^\alpha)'$ by taking the real part and considering terms in $\mathbb{C}[[u]]\tilde{z}^\alpha\bar{\tilde{z}}^\alpha$ in (4.18). Together with previous normalization, this amounts to setting

$$\mathcal{N}_{1010} = 0, \quad \mathcal{N}_{1211} = \ker \operatorname{Im} \mathbf{tr} \cap \ker \operatorname{Re}.$$

Thanks to the conditions $f_{\alpha 0}^\alpha(0) = 1$ for $|\alpha| = 1$, this enables us to determine uniquely the entire mapping (f, g) , as well as the right-hand side Ψ modulo the subspace $\mathcal{N}^n \subset \mathcal{F}^n$ of series, satisfying (1.3). The proposition is proved now. \square

Using, as in the one-dimensional case, the fact that any transformation (4.5) preserves the normalization conditions (1.3), we have:

Corollary 4.2. *The group of formal invertible transformations, preserving the germ at 0 of the model hypersurface (4.2), consists of the linear transformations (4.5).*

Corollary 4.3. *If $(N, 0)$ and $(\tilde{N}, 0)$ are two different normal forms of a fixed germ (M, p) , where $M \subset \mathbb{C}^{n+1}$ is a real-analytic hypersurface with generic Levi degeneracy at p , then there exists a linear transformation Λ , as in (4.5), which maps $(N, 0)$ into $(\tilde{N}, 0)$.*

4.2. Convergence of the normalizing transformation.

It remains to prove that the transformation, satisfying (4.6) and bringing $(M, 0)$ into a normal form, is a composition of certain (convergent) biholomorphic transformations. We describe these transformations below, giving a geometric interpretation for each of them.

Canonical pair of foliations on the Levi degeneracy set. We first define a pair of canonical foliations on the Levi degeneracy set $\Sigma \subset M$. Recall that $\Sigma \subset M$ is a codimension one submanifold, such that $T_p \Sigma \subset T_p M$ is transverse to the Levi kernel $K_p \subset T_p^\mathbb{C} M$ at every point $p \in \Sigma$. For a fixed point $p \in \Sigma$, let N denotes a (formal) normal form (1.3) of M at p , F a formal transformation, mapping (M, p) onto $(N, 0)$, and e^{n+1} the vector $(0, \dots, 0, 1) \in \mathbb{C}^{n+1}$. We then define a slope at p as follows:

$$l(p) := \text{span}_{\mathbb{R}} \{ (dF|_p)^{-1}(e^{n+1}) \}.$$

It follows from Corollary 4.3 that the definition of $l(p)$ is independent of the choice of a normal form and the corresponding normalizing transformation. Arguing identically to the two-dimensional case, we conclude that $l(p)$ depends on p analytically. We integrate $l(p)$ and obtain a (non-singular) real-analytic foliation \mathcal{T} of the real manifold Σ .

Definition 4.4. We call each of the leaves of the foliation \mathcal{T} a *degenerate chain*.

Each degenerate chain $\gamma \subset \Sigma$ at each point $p \in \gamma$ is transverse to the complex tangent $T_p^\mathbb{C} M$.

The second canonical foliation in Σ corresponds to the slope field

$$c(p) := K_p \cap T_p \Sigma, \quad p \in \Sigma.$$

Integrating $c(p)$ we obtain a real-analytic foliation \mathcal{S} of Σ . The canonical foliations \mathcal{T} and \mathcal{S} , called *the transverse* and *the tangent* foliations respectively, are transverse to each other and are biholomorphic invariants of $(M, 0)$.

Normalization of a chain and of the Levi kernels along the chain. For a germ $(M, 0)$ of a real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$, satisfying (4.1), with Levi degeneracy set Σ , we choose the unique degenerate chain $\gamma \subset \Sigma$, passing through 0. Let s_p denotes a leaf of the foliation \mathcal{S} , passing through a point $p \in \Sigma$. Consider then the set

$$S := \bigcup_{p \in \gamma} s_p.$$

Since the foliation \mathcal{S} is analytic and transverse to \mathcal{T} , by shrinking γ and the leaves s_p , $p \in \gamma$, we may assume that $S \subset \Sigma$ is a two-dimensional real-analytic submanifold. Moreover, $S \subset \mathbb{C}^{n+1}$ is totally real (since $l(p), c(p)$ lie in complementary complex subspaces of \mathbb{C}^{n+1}). Thus there exists a biholomorphic transformation $(\mathbb{C}^{n+1}, 0) \mapsto (\mathbb{C}^{n+1}, 0)$, transforming S into the totally real plane

$$\Pi^n = \{ \bar{z} = 0, \text{Re } z_n = 0, v = 0 \}, \quad (4.22)$$

γ into the line

$$\Gamma^n = \{ z = 0, v = 0 \} \subset \Pi^n, \quad (4.23)$$

and each s_p , $p \in \gamma$, into

$$\{\tilde{z} = 0, \operatorname{Re} z_n = 0, w = b\}, p = (0, \dots, 0, b), b \in \mathbb{R}. \quad (4.24)$$

It is not difficult to see that this transformation can be chosen in such a way that it preserves (4.1). The transformed hypersurface M^* contains Γ^n and thus satisfies, in addition to (4.1), the condition $\Phi_{00}^*(u) = 0$. In what follows we consider only transformations, preserving Γ^n and the conditions (4.1). It is *important to note* that in the normal form (1.3) the set S is not necessarily flat, as in (4.22). However, the slope field $c(p)$ remains horizontal for $p \in \Gamma^n$, as in (4.24), and the Levi kernels K_p , $p \in \Gamma^n$, all look as $\{\tilde{z} = 0, w = 0\}$.

Normalization of Segre varieties along a chain. Arguing identically to the corresponding step in Section 2, we perform the unique biholomorphic transformation of the form

$$z^* = z, \quad w^* = g(z, w), \quad g(0, w) = 0,$$

transforming $(M, 0)$ into a real-analytic hyperurface, satisfying $\Phi_{k'k00}^*(u) = 0$ for any $k', k \geq 0$. It is straightforward that Γ^n is preserved and the slope field $c(p)$ remains horizontal (i.e. of the form (4.24)) for $p \in \Gamma^n$. The Segre varieties of points $(\xi, \eta) \in \mathbb{C}^n \times \mathbb{C}$ with $(\xi, \eta) \in \Gamma^n$ are all of the form $\{w = \bar{\eta}\}$. In addition, we claim that for the new hypersurface M^* one has

$$\Phi_{z_j \bar{z}_n}(0, 0, u) = 0, \quad j = 1, \dots, n, \quad \text{i.e.} \quad \Phi_{1001} = \Phi_{0110} = \Phi_{0101} = 0. \quad (4.25)$$

This follows from the fact that the complex tangents $T_p^{\mathbb{C}} M$, $p \in \Gamma^n$, all have the form $\{w = 0\}$ in the new coordinates, while the Levi kernels K_p , $p \in \Gamma^n$, all remain $\{\tilde{z} = 0, w = 0\}$.

Normalization of the Segre map. This step can be interpreted as a normalization of the Segre map, considered as an antiholomorphic map

$$p \mapsto \left(w^p; w_{z_1}^p, \dots, w_{z_n}^p; w_{z_n^2}^p \right) \Big|_{z=0},$$

assigning to a point $p \in \mathbb{C}^{n+1}$ the partial 2-jet at $z = 0$ of its Segre variety $Q_p = \{w = w^p(z)\}$.

Similarly to Section 2, we introduce the subspace $\mathcal{D}^n \subset \mathcal{F}^n$, which consists of all convergent power series of the form

$$\sum_{k' \geq 2} (\Psi_{k'001}(\tilde{z}, \bar{\tilde{z}}, u) \bar{z}_n + \Psi_{01k'0}(\tilde{z}, \bar{\tilde{z}}, u) z_n) + \sum_{k'+k, l'+l \geq 2} \Psi_{k'kl'l}(\tilde{z}, \bar{\tilde{z}}, u) z_n^k \bar{z}_n^l.$$

We aim to bring a hypersurface M , obtained in the previous step, to such a form that all terms of weight ≥ 7 in Φ belong to the space \mathcal{D}^n , i.e., to bring the defining equation to the form

$$v = P(z, \bar{z}) \mod \mathcal{D}^n. \quad (4.26)$$

The subspace of terms to be removed from Φ in the current step is transverse to \mathcal{D}^n , and adding to Φ an element of \mathcal{D}^n does not change the desired form of it, that is why it is convenient for us to use identities modulo \mathcal{D}^n .

We begin by making the terms $\Phi_{1010}(\tilde{z}, \bar{\tilde{z}}, u)$ and $\Phi_{0201}(u)$ for the hypersurface M , obtained in the previous step, independent of u . For that we note that $\Phi_{1010}(\tilde{z}, \bar{\tilde{z}}, u)$ is an analytic family $H_u(\tilde{z}, \bar{\tilde{z}})$ of Hermitian forms, and $H_0(\tilde{z}, \bar{\tilde{z}}) = \langle \tilde{z}, \bar{\tilde{z}} \rangle$ is nondegenerate. We denote the analytic function $\Phi_{0201}(u)$ as $a(u)$, $a(0) = 1$. Then by the implicit function theorem, there exist real-analytic functions $T(u) \in \operatorname{GL}(n-1, \mathbb{C})$ and $c(u) \in \mathbb{C}^*$ near the origin such that

$$H_u(\tilde{z}, \bar{\tilde{z}}) = \langle T(u)\tilde{z}, \overline{T(u)\tilde{z}} \rangle, \quad a(u) = (c(u))^2 \overline{c(u)}, \quad T(0) = \operatorname{Id}, \quad c(0) = 1.$$

Then we perform the biholomorphic transformation

$$\tilde{z} \mapsto T(w)\tilde{z}, \quad z_n \mapsto c(w)z_n, \quad w \mapsto w. \quad (4.27)$$

Since for the initial hypersurface the defining function $\Phi(z, \bar{z}, u)$ contains only terms of the form $\Phi_{k'kl'l}$, $k' + k \geq 1$, $l' + l \geq 1$, the same property holds for the new defining function Φ^* of the new hypersurface M^* , and we compute

$$\Phi(z, \bar{z}, u) = \Phi^*(z^*, \bar{z}^*, u^*) = \Phi^*(\check{z}T(u), c(u)z_n, \overline{\check{z}T(u)}, \overline{c(u)z_n}, u) \quad \text{mod } \mathcal{D}^n,$$

provided $(z, w) \in M$. It follows from here that

$$\Phi_{1010}(\check{z}, \bar{\check{z}}, u) = \Phi_{1010}^*(\check{z}T(u), \overline{\check{z}T(u)}, u)$$

and

$$\Phi_{0201}(u) = \Phi_{0201}^*(u)(c(u))^2 \overline{c(u)},$$

so that

$$\Phi_{1010}^*(\check{z}^*, \bar{\check{z}}^*, u^*) = \langle \check{z}^*, \bar{\check{z}}^* \rangle, \quad \Phi_{0201}^*(u^*) = 1,$$

as required.

We then achieve the condition (4.26) by performing a transformation

$$z^* = z + f(z, w), \quad w^* = w, \quad f(z, w) = O(|z|^2),$$

where f will be determined below. We do so in several steps.

First, we eliminate terms $\Phi_{k'k10}$ with $k' + k \geq 2$ by a transformation

$$(\check{z}, z_n, w) \mapsto (\check{z} + \check{f}(z, w), z_n, w), \quad \check{f}(z, w) = O(|z|^2).$$

For that, we choose a holomorphic function $B(z, u) = O(|z|^2) \in \mathbb{C}^{n-1}$ satisfying

$$\sum_{k'+k \geq 2} \Phi_{k'k10}(\check{z}, \bar{\check{z}}, u) z_n^k = \langle B(z, u), \bar{\check{z}} \rangle.$$

Then, arguing as above and using identities modulo \mathcal{D}^n , we see that the transformation

$$(\check{z}, z_n, w) \mapsto (\check{z} + B(z, u), z_n, w)$$

transforms M into a hypersurface satisfying

$$\Phi_{k'k10}^* = 0, \quad k' + k \geq 2.$$

Second, we use similar arguments to remove the terms $\Phi_{k'101}$ with $k' \geq 1$ by a transformation

$$(\check{z}, z^n, w) \mapsto (\check{z}, z^n + f^n(\check{z}, w), w), \quad f^n = O(|\check{z}|^2).$$

Thanks to (4.25), we expand the defining function of the hypersurface, obtained in the previous step, as

$$\begin{aligned} \Phi(z, \bar{z}, u) &= \langle \check{z}, \bar{\check{z}} \rangle + (z^n + \varphi(\check{z}, u))^2 \overline{(z^n + \varphi(\check{z}, u))} + (z^n + \varphi(\check{z}, u)) \overline{(z^n + \varphi(\check{z}, u))^2} + \\ &\quad + (z^n)^2 \overline{z^n} \cdot O(|z|) + z^n (\overline{z^n})^2 \cdot O(|z|) \quad (\text{mod } \mathcal{D}^n) \end{aligned}$$

for an appropriate analytic function $\varphi(\check{z}, u) = O(|\check{z}|^2)$ (more precisely, we choose $\varphi := \frac{1}{2} \sum_{k' \geq 2} \Phi_{k'101}$). Then putting $f^n := \varphi(\check{z}, w)$, we obtain the desired transformation.

Finally, we remove the terms Φ_{kl01} , $l \geq 2$, $k + l \geq 3$. We expand the defining function of the hypersurface, obtained in the previous step, as

$$\begin{aligned} \Phi(z, \bar{z}, u) &= \langle \check{z}, \bar{\check{z}} \rangle + (z_n + z_n \psi(z, u))^2 \overline{(z_n + z_n \psi(z, u))} + \\ &\quad + (z_n + z_n \psi(z, u)) \overline{(z_n + z_n \psi(z, u))^2} \quad \text{mod } \mathcal{D}^n \end{aligned}$$

for an appropriate real-analytic function $\psi(z, u) = O(|z|)$, where one determines ψ by the implicit function theorem from the equation

$$2\psi + \psi^2 = \sum_{k+l \geq 3, l \geq 2} \Phi_{kl01}(\tilde{z}, 0, u) z_n^{l-2}.$$

Then we achieve (4.26) by performing the transformation

$$(\tilde{z}, z_n, w) \mapsto (\tilde{z}, z_n + z_n \psi(z, w), w).$$

The Segre map is now given by

$$s = (\xi, \eta) \mapsto (\bar{\eta}; 2i\varepsilon^1 \bar{\xi}_1, \dots, 2i\varepsilon^{n-1} \bar{\xi}_{n-1}, 2i(\bar{\xi}^n)^2 + O(|\tilde{\xi}|^2); 4i\bar{\xi}^n + O(|\xi|^2)). \quad (4.28)$$

Clearly, Γ^n is preserved and the Levi kernels K_p , $p \in \Gamma^n$, all remain of the form $\{\tilde{z} = 0, w = 0\}$.

Fixing an orthonormal basis in the Levi-nondegenerate direction along a chain. We next achieve the normalization condition

$$\operatorname{Re} \Phi_{1211} = 0$$

by means of a transformation

$$(\tilde{z}, z_n, w) \mapsto (C(w)\tilde{z}, z_n, w) \quad (4.29)$$

for an appropriate holomorphic near the origin function $C(w)$ such that $C(w) \in U(r, n-1-r)$ for $w \in \mathbb{R}$ and $C(0) = \operatorname{Id}$. This step can be interpreted as an analytic choice of an orthonormal basis $\{e^1(u_0), \dots, e^{n-1}(u_0)\}$ with respect to the form $\langle \tilde{z}, \bar{\tilde{z}} \rangle$ in the Levi-nondegenerate direction K_p^T at every point $p = (0, u_0) \in \Gamma^n$.

Let us introduce the subspace $\mathcal{C}^n \subset \mathcal{D}^n$ (where $\mathcal{D}^n \subset \mathcal{F}^n$ is the space of power series used in the previous step), which consists of elements $\Phi \in \mathcal{D}^n$ satisfying $\Phi_{1211} = 0$. It is again convenient for us to use identities modulo \mathcal{C}^n . Let us then fix some analytic function $C(u)$, valued in $U(r, n-1-r)$. Note that for any fixed $a, b \in \mathbb{C}^{n-1}$ we have

$$0 = \left(\langle C(u)a, \overline{C(u)b} \rangle \right)' = \langle C'(u)a, \overline{C(u)b} \rangle + \langle C(u)a, \overline{C'(u)b} \rangle.$$

Thus, after the transformation (4.29), we obtain (recall that $P = P(z, \bar{z})$ denotes the leading polynomial of Φ)

$$\begin{aligned} \Phi(z, \bar{z}, u) &= \Phi^*(z^*, \bar{z}^*, u^*) = 2\operatorname{Re}(z_n^2 \bar{z}_n) + 2\operatorname{Re}(\Phi_{1211}^* z_n^2 \bar{z}_n) + \\ &\quad + \langle (C(u) + iC'(u)P)\tilde{z}, \overline{(C(u) + iC'(u)P)\tilde{z}} \rangle \mod \mathcal{C}^n = \\ &= P(z, \bar{z}) + 2\operatorname{Re}(\Phi_{1211}^* z_n^2 \bar{z}_n) + P\langle iC'\tilde{z}, \overline{C\tilde{z}} \rangle + P\langle C\tilde{z}, -i\overline{C'\tilde{z}} \rangle \mod \mathcal{C}^n = \\ &= P(z, \bar{z}) + 2\operatorname{Re}(\Phi_{1211}^* z_n^2 \bar{z}_n) + 2(z_n^2 \bar{z}_n + z_n(\bar{z}_n)^2)\langle iC'\tilde{z}, \overline{C\tilde{z}} \rangle \mod \mathcal{C}^n, \end{aligned}$$

provided $(z, w) \in M$. Thus we have

$$\Phi_{1211}(\tilde{z}, \bar{\tilde{z}}, u) = \Phi_{1211}^*(\tilde{z}, \bar{\tilde{z}}, u) + 2\langle iC'\tilde{z}, \overline{C\tilde{z}} \rangle,$$

and the condition $\operatorname{Re} \Phi_{1211}^* = 0$ amounts to representing the Hermitian form $M(\tilde{z}, \bar{\tilde{z}}) := \frac{1}{2}\operatorname{Re} \Phi_{1211}$ in the form $\langle iC'\tilde{z}, \overline{C\tilde{z}} \rangle$ for an appropriate function $C(u) \in U(r, n-1-r)$. Since the Hermitian form $\langle \cdot, \cdot \rangle$ is nondegenerate, the condition

$$M(\tilde{z}, \bar{\tilde{z}}) = \langle iC'\tilde{z}, \overline{C\tilde{z}} \rangle$$

can be read as a first order nonsingular analytic ODE for $C(u) \in \operatorname{Mat}(n-1, \mathbb{C})$ near the point $u = 0, C = \operatorname{Id}$, that we solve with the initial condition $C(0) = \operatorname{Id}$. It remains to show that for

$C(u)$ chosen in this manner we indeed have $C(u) \in U(r, n-1-r)$. This can be argued as follows. As $M(\tilde{z}, \bar{\tilde{z}})$ is a Hermitian form, we have

$$\langle C'a, \overline{Cb} \rangle + \langle Ca, \overline{C'b} \rangle = 0 \quad \forall a, b \in \mathbb{C}^{n-1},$$

thus

$$\left(\langle Ca, \overline{Cb} \rangle \right)' = \langle C'a, \overline{Cb} \rangle + \langle Ca, \overline{C'b} \rangle = 0$$

for any fixed $a, b \in \mathbb{C}^{n-1}$, so that

$$\langle C(u)a, \overline{C(u)b} \rangle = \langle C(0)a, \overline{C(0)b} \rangle = \langle a, \bar{b} \rangle,$$

as required.

Fixing a parametrization for a chain. For the hypersurface, obtained in the previous step, we claim that

$$\Phi_{1102}(\tilde{z}, \bar{\tilde{z}}, u) = \mathbf{tr} \Phi_{1111}(\tilde{z}, \bar{\tilde{z}}, u) = 0$$

in (4.26).

Indeed, consider the (formal) transformation $F = (f, g) = (\check{f}, f^n, g)$, bringing a hypersurface (4.26) into normal form, and study the equation (4.4), applied to it. Collecting terms with $z_n \bar{z}_n u^l$, $l \geq 0$, we first obtain $\operatorname{Re} f_{00}^n(u) = 0$. Collecting all terms of the form $\check{z}^\alpha (\bar{\check{z}})^\beta z_n \bar{z}_n u^0$, $|\alpha| = |\beta| = 1$, we obtain precisely the equation (4.11), where we substitute $u := 0$ and $\Psi := 2\Phi$. Since $\Gamma^n \subset M$ is a degenerate chain, we have $(f_{00}^n)'(0) = 0$, so that $\mathbf{tr} \Phi_{1111}(\tilde{z}, \bar{\tilde{z}}, 0) = 0$.

Next, collecting first in (4.4) all terms of the form $\check{z}^\alpha u$, $|\alpha| = 1$, and second all terms of the form $\check{z}^\alpha \bar{z}_n (z_n)^2 u^0$, $|\alpha| = 1$, it is straightforward to check that we obtain precisely the equations (4.13), where the first equation is differentiated and substituted $u := 0$, $\Psi_{1000} := 0$, and the second is substituted $u := 0$, $\Psi_{1102} := 2\Phi_{1102}$. Thanks to the fact that Γ^n is a degenerate chain we have $\check{f}'_{00}(0) = 0$, and we conclude that $\Phi_{1102}(\tilde{z}, \bar{\tilde{z}}, 0) = 0$. Since the prenormal form (4.26) is invariant under the real shifts $w \mapsto w + u_0$, $u_0 \in \mathbb{R}$, and $\Gamma^n \subset M$ is a degenerate chain, we similarly conclude that

$$\Phi_{1102}(\tilde{z}, \bar{\tilde{z}}, u_0) = \mathbf{tr} \Phi_{1111}(\tilde{z}, \bar{\tilde{z}}, u_0) = 0$$

for any small $u_0 \in \mathbb{R}$ in (4.26), as required.

It remains to achieve the last normalization condition

$$\operatorname{Im} \mathbf{tr} \Phi_{1211} = 0,$$

using, as before, an appropriate gauge transformation

$$\tilde{z} \mapsto \check{f}(w)\tilde{z}, \quad z_n \mapsto f^n(w)z_n, \quad w \mapsto g(w), \quad \check{f}(0) \neq 0, f^n(0) \neq 0, g(0) = 0, g'(0) \neq 0. \quad (4.30)$$

We do so by choosing a gauge transformation

$$\tilde{z}^* = (q'(u))^{1/2} \tilde{z}, \quad z_n^* = (q'(w))^{1/3} z_n, \quad w^* = q(w)$$

for an appropriate $q(w)$ with $\operatorname{Im} q(u) = 0$, $q(0) = 0$, $q'(0) > 0$. This transformation can be interpreted as a choice of parametrization for the initial degenerate chain γ .

We introduce the subspace $\mathcal{A}^n \subset \mathcal{C}^n$, characterized by the conditions

$$\Phi_{1102} = \Phi_{1111} = 0.$$

For the same reason as in the previous steps, we use identities modulo \mathcal{A}^n . Note also that \mathcal{A}^n is contained in the space \mathcal{N}^n of series, satisfying the normalization conditions (1.3).

Applying the above transformation to a hypersurface, satisfying (4.26) and

$$\Phi_{1102} = \operatorname{Re} \Phi_{1211} = \mathbf{tr} \Phi_{1111} = 0,$$

it is straightforward to compute (as before, we use expansions of the form $h(u + iv) = h(u) + ih'(u)v + \dots$)

$$\begin{aligned} v^* &= q'(u)v \mod \mathcal{A}^n, \\ P(z^*, \bar{z}^*) &= q'(u)P(z, \bar{z}) + \frac{i}{3}q''(u)\langle \tilde{z}, \bar{\tilde{z}} \rangle (z_n)^2 \bar{z}_n \mod \mathcal{A}^n \\ v^* - P(z^*, \bar{z}^*) &= q'(u)(v - P(z, \bar{z})) + \frac{i}{3}q''(u)\langle \tilde{z}, \bar{\tilde{z}} \rangle ((z_n)^2 \bar{z}_n - z_n (\bar{z}_n)^2) \mod \mathcal{A}^n. \end{aligned}$$

Hence

$$\operatorname{Im} \operatorname{tr} \Phi_{1211}^* = q' \cdot \operatorname{Im} \operatorname{tr} \Phi_{1211} + \frac{2(n-1)}{3} q'',$$

so that the condition $\operatorname{Im} \operatorname{tr} \Phi_{1211}^* = 0$ becomes a second order nonsingular ODE for $q(u)$. Solving it with some initial condition $q'(0) > 0$, we finally obtain a hypersurface, satisfying all the normalization conditions (1.3). It is not difficult to see that the gauge transformation chosen to achieve $\operatorname{Im} \operatorname{tr} \Phi_{1211}^* = 0$ must have the above form and hence is unique up to the choice of the real parameter $q'(0)$, corresponding to the action of the subgroup

$$\tilde{z} \mapsto \lambda^3 \tilde{z}, \quad z_n \mapsto \lambda^2 z_n, \quad w \mapsto \lambda^6 w. \quad (4.31)$$

in (4.5). Thus

we can canonically, up to the action of the group (4.31), choose a parametrization on each degenerate chain.

Theorem 2 is completely proved now. As in the two-dimensional case, we can see from the proof that in the normal form (1.3) the unique degenerate chain, passing through the origin, is given by (4.23).

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